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Hamiltonian Dynamics with External Forces and Observations

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Abstract. In this paper a definition of a (nonlinear) Hamiltonian system with inputs and outputs is given, which generalizes both the definition of a linear Hamiltonian system with inputs and outputs and the differential geometric definition of a Hamiltonian vectorfield. Specialized to the case of Lagrangian systems this definition generates the Euler-Lagrange equations with external forces. Further interconnections of Hamiltonian systems are treated and the close relationship with network theory is showed. Finally the newly developed theory is applied to the study of symmetries and to a realization theory for Hamiltonian systems. It will be argued that this way of describing Hamiltonian systems can be extended to a broader class of physical systems.

1. Introduction

1. In most mathematical textbooks on classical mechanics nowadays great emphasis is put on what is called *analytical* mechanics, i.e. mechanical systems which can be described without external influences, such as the solar system. They consider mechanical systems without forces; when forces are present (such as gravitation) they are assumed as coming from a potential field. By incorporating this potential in the system the enlarged system can be treated as a system without external forces. Although this operation from (conservative) forces to a potential function already seems to deserve careful attention, the idea of force as such is used only as a remainder from physics and is never fully exploited. Nevertheless, when we look at the older works on mechanics (see for instance [1, 2]), the concept of force is treated as one of the basic concepts of mechanics. Also from a practical point of view the possibility of exerting forces on a system is very basic and consequently in the more technical literature forces still have a very important place.

Modern mathematics has given a very elegant set-up for the study of mechanical systems without external forces (see the books of Arnold [3] and Abraham & Marsden [4]). On the other side, mathematical systems theory provides an adequate conceptual framework for treating systems with inputs and outputs. For describing mechanical systems *with* external influences it seems

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desirable to bring these approaches together. Already Brockett [5] has given an exposé on how systems theory and mechanics might work together. In [6] Takens has given, from a different viewpoint, a model for describing mechanical, i.e. Lagrangian, systems with external forces. As is evident from these two references; it is not altogether clear how to formalize mechanical systems with external influences and partial observation of the state of the system. Willems ([7, 8]) has proposed a system theoretic framework which is more general and seems more useful than the usual input-output framework for treating physical systems. Although the input-output setting is very natural in the context of control theory, in the description of physical systems (where "physical" is interpreted in a broad sense) it is often not clear which of the external variables are the inputs and which are the outputs. Consider for instance an electrical network; one can sometimes describe it with the voltages as inputs and the currents as outputs, or *vice versa*. Therefore it seems desirable not to identify a priori which external variables constitute causes (inputs) and which constitute effects (outputs). This can itself be regarded as a modelling question.

2. We now give a typical example of a mechanical system, which illustrates and motivates the set-up of the following sections.

Consider a pointmass m with position q_1, influenced by a force F_1. According to Newton's second law the relation between q_1 and F_1 as functions of time t is given by

$$m\ddot{q}_1 = F_1. \tag{1.1}$$

We consider (1.1) as a state-space description of a mechanical system with input F_1 and output equal to the position q_1. Note that we see F_1 as a *basic* variable and that (1.1) really expresses a compatibility relation between the input functions $F_1(t)$ and the output functions $q_1(t)$.

Next we look at another mechanical system. Take a potential function $V(q_2)$. This defines a force F_2 as follows:

$$F_2 = -\frac{dV}{dq}(q_2). \tag{1.2}$$

One can view this as a (memoryless) mechanical system which relates the variables $F_2(t)$ and $q_2(t)$. Note that it is now natural to see the position q_2 as the input and F_2 as the output.

The mechanical systems (1.1) and (1.2) can be *interconnected* by setting

$$q_1 = q_2, \qquad F_1 = F_2. \tag{1.3}$$

(this can be regarded as Newton's third law).

The resulting system has the form (setting $q = q_1 = q_2$)

$$m\ddot{q} + \frac{dV}{dq}(q) = 0. \tag{1.4}$$

This is a mechanical system without inputs. As outputs we could take the position q, or the position q together with $-\frac{dV}{dq}(q)$, which is now the *internal* force. Also the interconnection (1.3) has a special form. If we consider the space

$$W = \{(q_1, F_2, F_1, q_2) \mid q_i \in \mathbb{R}, F_i \in \mathbb{R}, i=1,2\} = \mathbb{R}^4$$

with the natural symplectic form on it then the interconnection defines a lagrangian subspace of W (see for definitions § 3). With the aid of this we can see (1.3) as a memoryless mechanical system. Because the interconnection here is nonmixing (i.e. q_1 is only related to q_2, F_1 only to F_2) we get the special form which is intimately related to Kirchhoff's laws (see § 6).

3. The paper consists of the following parts. In § 2 we consider the general formulation of systems with external variables. In § 3 we give a short review of Hamiltonian mechanics such as treated in Arnold [3] and Abraham & Marsden [4]. In § 4 we give our definition of a Hamiltonian system in state space form with external variables. In § 5 we define as a special but important case Lagrangian systems. The Euler-Lagrange equations appear to fit very nicely in this framework. Interconnections of Hamiltonian and Lagrangian systems are treated in § 6. In § 7 we look at a realization theory for Hamiltonian systems. The linear case will be worked out. In § 8 we consider in our set-up the formulation of symmetries, a well known and important subject in the case without external variables. § 9 contains the conclusion.

2. Differential Systems in State Space Form

In this § we will give some general definitions of systems with external variables, which will be used in the following §§ in the special case of Hamiltonian systems. First we will give definitions for the very general case of systems without differential structure which were already stated in Willems [7].

Definition 2.1. Let W be a set, called the signal alphabet. Let T denote the time axis (mostly $\mathbb{R}$ or $\mathbb{Z}$). An external dynamical system Σ_e (on W) is a subset of W^T.

So an external dynamical system simply consists of a set of functions from the time axis to W, the set of external variables (usually not finite).

In the usual input-output framework we have an input alphabet U and a output alphabet Y, and moreover there is a function F from the set of input functions U^T to the set of output functions Y^T (F is called the input-output map). If we now define $W = U \times Y$ then it is easily seen that the set $\{(u(\cdot), (Fu)(\cdot)) \mid u(\cdot): T \to U\} \subset W^T$ defines a system in the sense of def. 2.1.

In our more general case we do not assume ab initio that we can split the external variables in a set of inputs and a set of outputs, which are causally related to the inputs. We merely assume that not all functions of the time axis to W occur, i.e. we assume that there are dynamical compatibility relations between the external variables.

In most situations we are not satisfied with such a description of a system only in external terms. We want to have a "model" of the system which in some sense explains, through some internal dynamical mechanism, the external behavior of the system. Moreover we may hope that such a model gives a better insight in the structure of the system. Therefore we introduce a set X of variables, called the state variables, who can be thought of expressing the memory (state) of the system. Following [7] we can do this as follows. Let R be a relation on (i.e. a subset of) $A \times B$. We define R_A as the projection of R on A, and R_B the projection of R on B. Then the relation R is said to be a *product* relation if $R = R_A \times R_B$. Now let R be a relation on (i.e. a subset of) $A \times B \times C$. We will call $R_{\{x_2=b\}} := \{(x_1, x_3) | (x_1, b, x_3) \in R\}$ the relation R *conditioned* by $\{x_2 = b\}$. We will say that x_1 and x_3 are *independent* given x_2, or equivalently that x_2 *splits* x_1 and x_3 if, for all $b \in B$, $R_{\{x_2=b\}}$ is a product relation on $A \times C$.

These notions are easily generalized to relations on sets of the form $\prod_{\gamma \in \Gamma} A_\gamma$ with Γ an arbitrary index set. In fact, we can now give the definition of a system in state space form.

Definition 2.2. Let W and T be as before (think of T as $\mathbb{R}$), and let X be a set, called the *state space*. A dynamical system in state space form is defined as a system Σ on $X \times W$ which satisfies the axiom:

$$(A): x(t) \text{ splits } \{x(\tau), w(\tau); \tau < t\} \text{ and } \{x(\tau), w(\tau); \tau \geq t\}.$$

Explicitly, if $(x_i, w_i) \in \Sigma$, $i = 1, 2$, and $x_1(t) = x_2(t)$, then also $(x, w) \in \Sigma$ with (x, w) defined by

$$(x, w)(\tau) := \begin{cases} (x_1, w_1)(\tau) & \text{for } \tau < t \\ (x_2, w_2)(\tau) & \text{for } \tau \geq t \end{cases}.$$

A system in state space form defines in a natural way a system on W as follows:

Definition 2.3. The system $\Sigma_e := \{w | \exists x \text{ such that } (x, w) \in \Sigma\}$ is called the external behavior of Σ and Σ is called a (state space) *representation* (or *realization*) of Σ_e.

Now we turn to the definition which shall be used mainly in the following §§. We assume a certain smoothness of our systems. Then the following definition is very appealing.

Definition 2.4. A differential system in state space form is described by:

(i) smooth manifolds X, W (smooth always means C^∞).

(ii) a smooth bundle $\pi: B \to X$.

(iii) a smooth map f: $B \to TX \times W$ for which the diagram

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "\pi"'] & & TX \times W \arrow[dl, "\pi_X"] \\ & X & \end{tikzcd}$$

commutes.

Now the system itself is defined by $\Sigma := \{(x,w)\colon \mathbf{R} \to X \times W \mid x$ absolute continuous and $(\dot{x}(t), w(t)) \in f(\pi^{-1}(x(t)))$ a.e.$\}$ and will be denoted by $\Sigma(X, W, B, f)$. It is useful to think of $\pi^{-1}(x)$ as the input space when the system is in state x. Note that this input space is dependent on the state x. Only when the bundle $\pi\colon B \to X$ is *trivial*, i.e. $B = X \times U$ for a manifold U this is not the case. When we think of forces working in a point $x \in X$ this state dependence of the inputspace is very natural. The case of only one vectorfield on a manifold X is easily recovered as can be seen from the following definition.

Definition 2.5. An autonomous differential system in state space form (i.e., autonomous interpreted as without inputs) is a system $\Sigma(X, W, B, f)$ where the fibres of B consist of only one point.

If we denote the local coordinates of X by x, the local coordinates of W by w and the local coordinates of the fibres of B by u we could equivalently describe our system as

$$\dot{x} = g(x,u), \qquad w = h(x,u). \qquad [2.1]$$

where g and h are smooth maps.

Of course, in most situations we would like to have that B is a "nice" subbundle of $TX \times W$. The following assumptions have a good interpretation:

(i) the matrix $\left(\dfrac{\partial g}{\partial u}\right)$ has maximal rank.

(ii) the matrix $\left(\dfrac{\partial h}{\partial u}\right)$ has maximal rank.

Condition (i) implies that $f(B) \subset TX \times W$ projected on TX is a regular subbundle of TX. So there are no "singularities". Condition (ii) implies that locally we can "solve" for the inputs. The implicit function theorem tells us that locally we can find coordinates $w = (w_1, w_2)$ for W and a map $\tilde{h}$ such that $w = h(x,u)$ is replaced by

$$w_2 = \tilde{h}(x, w_1)$$

i.e., we can interpret some external variables as (local) inputs! However, to cover all situations, we should relax our assumptions somewhat and assume only that f: $B \to TX \times W$ is an injective immersion. To avoid technical difficulties, we will make in the following §§ the somewhat stronger assumption that f(B) is a (regular) submanifold of $TX \times W$.

In terms of definition 2.4, we can define the notions of feedback equivalence and of minimality.

Definition 2.6. The systems $\Sigma_1(X, W, B, f)$ and $\Sigma_2(X, W, B', f')$ are *feedback equivalent* if there exists a bundle diffeomorphism $F\colon B \to B'$ such that the diagram

$$\begin{tikzcd}
B' \arrow[rr, bend left, "f'"] \arrow[dr, "\pi'"'] & B \arrow[l, "F"'] \arrow[r, "f"] \arrow[d, "\pi"] & TX \times W \arrow[dl, "\pi_X"] \\
 & X &
\end{tikzcd}$$

commutes.

When constructing a realization for an external system, it is often desirable to keep the state space as small as possible. This *minimality* of the realization is defined as follows (see also [7]).

Definition 2.7. Let $\Sigma(X, W, B, f)$ and $\Sigma'(X', W, B', f')$ be two differential systems. Then we say $\Sigma' \leq \Sigma$ if there exist surjective submersions $\varphi\colon X \to X'$, $\Phi\colon B \to B'$ such that the diagram

$$\begin{tikzcd}
B \arrow[rrr, "\Phi"] \arrow[dr, "f"] \arrow[dddd, "\pi"'] & & & B' \arrow[dl, "f'"'] \arrow[dddd, "\pi'"] \\
 & W \arrow[r, "id"] & W & \\
 & \times & \times & \\
 & TX \arrow[r, "\varphi_*"] \arrow[dl, "\pi_X"] & TX' \arrow[dr, "\pi_{X'}"'] & \\
X \arrow[rrr, "\varphi"'] & & & X'
\end{tikzcd}$$

commutes.

Σ is called *equivalent* with Σ' (denoted $\Sigma \sim \Sigma'$) if φ and Φ are diffeomorphisms. We call Σ *minimal* if: $\Sigma' \leq \Sigma \Rightarrow \Sigma' \sim \Sigma$.

3. A Short Review of Hamiltonian Mechanics

In this paragraph we will review very briefly the main notions of Hamiltonian mechanics which will be used in the following §§. Most of this is treated in great detail in [3] or [4].

1. Basic to the description of Hamiltonian systems is a smooth manifold M (think of the phase space) with a 2-form ω which satisfies:

(i) ω is nondegenerate; i.e., for every $0 \neq X \in T_xM$ there exists $Y \in T_xM$ such that $\omega_x(X, Y) \neq 0$.

(ii) ω is a closed 2-form; i.e., $d\omega = 0$. ω is called a symplectic form.

From (i) it follows that M is necessarily an even dimensional manifold. From (i) and (ii) it can be deduced that there are local coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ for

M (we take M 2n-dimensional) such that

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i. \qquad [3.1]$$

This is known as Darboux's theorem. (M, ω) is called a symplectic manifold.

A vectorfield $X: M \to TM$ is a locally *Hamiltonian* vectorfield iff $L_X\omega = 0$, where $L_X\omega$ is the Lie derivative of ω with respect to X (most times we will drop the prefix "locally"). From the formulation $L_X\omega = i_X d\omega + d i_X\omega$ (with $i_X\alpha(Y_1, \ldots, Y_k) := \alpha(X, Y_1, \ldots, Y_k)$ for a $(k+1)$-form α) and the closedness of ω it follows that $L_X\omega = 0$ is equivalent to $d i_X\omega = 0$. Then Poincaré's lemma gives that, at least locally, there exists a function $H: M \to \mathbb{R}$ such that

$$i_X\omega = dH. \qquad [3.2]$$

If we write this out in the local coordinates of expression (3.1), we obtain from (3.2) the usual expressions for a Hamiltonian vectorfield:

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \end{cases} \quad i = 1, \cdots, n. \qquad [3.3]$$

The most typical example of a symplectic manifold is a cotangent bundle. Let T^*Q be a cotangent bundle (think of Q as the configuration space), then we can define a natural 1-form θ on T^*Q as follows:

Take $\alpha \in T^*Q, X \in T_\alpha(T^*Q)$, and let π denote the projection of T^*Q on Q. Then we define

$$\theta_\alpha(X) = \alpha\big(\pi_*(X)\big) \qquad [3.4]$$

In local coordinates $\theta = \sum_i p_i dq_i$, where q_i are local coordinates for Q, and p_i the coordinates of the fibres of T^*Q. θ is called the canonical 1-form on T^*Q.

It is easy to check that $\omega := -d\theta$ is a symplectic form on T^*Q which in local coordinates is equal to $\sum_i dq_i \wedge dp_i$.

Instead of the definition of a Hamiltonian vectorfield in terms of $L_X\omega = 0$, we will take another equivalent point of view which is more appropriate to our framework. For this we need the notion of a lagrangian submanifold.

Definition 3.1. A submanifold N of a symplectic manifold (M, ω) is called lagrangian if:

(i) $\omega(X, Y) = 0 \ \forall X, Y \in TN$.

(ii) N has maximal dimension; it can be deduced from (i) that the maximal dimension is half the dimension of M.

Related are the definitions of isotropic and co-isotropic submanifolds.

Definition 3.2.
a. A submanifold N of (M, ω) is called *isotropic* if $\omega(X,Y)=0\ \forall X,Y \in TN$ (so a lagrangian submanifold is also an isotropic submanifold).
b. A submanifold N of (M, ω) is called *co-isotropic* if $(TN)^{\perp} \subset TN$, where $(TN)^{\perp}$ is the orthogonal complement of TN with respect to ω, i.e. for $x \in N$

$$(T_xN)^{\perp} := \{X \in T_xM \mid \omega(X,Y)=0, \forall Y \in T_xN\}.$$

In this language N is isotropic amounts to saying that $TN \subset (TN)^{\perp}$ and N is lagrangian is equivalent with $TN=(TN)^{\perp}$.

Now we can given an alternative, but equivalent definition of a Hamiltonian vectorfield as follows (see Tulczyjew [9] and Hermann [10]). Let (M, ω) be a symplectic manifold. Because ω is nondegenerate it defines a bundle isomorphism $\bar{\omega}$ from TM to T*M by setting

$$\bar{\omega}(X) := \omega(X, -) \quad \text{for } X \in TM. \qquad [3.5]$$

T*M is a cotangentbundle, so it has the 1-form θ defined above (3.4). Then $\bar{\omega}^*\theta$ is a 1-form on TM and $d\bar{\omega}^*\theta$ is a symplectic form on TM which we shall denote by $\dot{\omega}$. We can calculate $\dot{\omega}$ in local coordinates as follows. Take local coordinates $q_1, \cdots, q_n, p_1, \cdots, p_n$ for M such that $\omega = \sum_i dq_i \wedge dp_i$.

Now we can construct coordinates for TM. Let f: $M \to \mathbb{R}$ then define f: $TM \to \mathbb{R}$ as

$$\dot{f}(v_p) = df_p(v_p) \qquad [3.6]$$

with $v_p \in TM$ ($\pi(v_p)=p \in M$, π projection of TM on M). Then one can see that $(q_1, \cdots, q_n, p_1, \cdots, p_n, \dot{q}_1, \cdots, \dot{q}_n, \dot{p}_1, \cdots, \dot{p}_n)$ forms a coordinate system for TM, and that $\dot{\omega}$ defined above has in these coordinates the form

$$\dot{\omega} = \Sigma_i d\dot{q}_i \wedge dp_i + dq_i \wedge d\dot{p}_i \qquad [3.7]$$

(which explains the notation $\dot{\omega}$).

Now we can give

Definition 3.3. A vectorfield X: $M \to TM$ is called Hamiltonian if the graph of X is a lagrangian submanifold of $(TM, \dot{\omega})$

Remark. One can check that this definition is equivalent to $L_X\omega=0$ (see [9]).

2. A special but important case of a Hamiltonian vectorfield is, what we shall call a little misleadingly a *Lagrangian* vectorfield. In this case we must start with a cotangentbundle $M=T^*Q$ (Q is the configurationspace) with the 1-form θ defined by (3.4). Analogous to the construction of $\dot{\omega}$ we can now define a 1-form denoted by $\dot{\theta}$ on T(T*Q) as follows (see [9]). θ is a 1-form on T*Q, so we can also see θ as a function on T(T*Q) which we, for clearness' sake denote by the notation $\hat{\theta}$. So $\hat{\theta}(X) := \theta(X)$ for $X \in T(T^*Q)$. Furthermore, when we have an arbitrary manifold K

then on TTK there is defined a natural involution (see [9]). If $(x,\dot{x})$ are coordinates for TK then we denote coordinates for T(TK) by $(x,\dot{x},\delta x,\delta\dot{x})$. Then the involution $\sim$ is given by

$$\sim : (x,\dot{x},\delta x,\delta\dot{x}) \mapsto (x,\delta x,\dot{x},\delta\dot{x}). \tag{3.8}$$

Now we define $\dot{\theta}$ by

$$\dot{\theta}(X) := \tilde{X}(\hat{\theta}) \text{ for } X \in TT(T^*Q). \tag{3.9}$$

In local coordinates $\dot{\theta}$ is given by (q_i, p_i are coordinates for T*Q), $\dot{\theta} = \sum_i \dot{p}_i dq_i + p_i d\dot{q}_i$ (see [9]).

Definition 3.4 (see [9]). A vectorfield X: $T^*Q \to T(T^*Q)$ is a Lagrangian vectorfield if there exists a function L: $TQ \to \mathbb{R}$ such that $\dot{\theta}$ restricted to the graph of X in $T(T^*Q)$ is equal to dL.

Remark. So the graph of X is a lagrangian submanifold of $(T(T^*Q), d\dot{\theta})$.
In local coordinates def. 3.4 says that

$$\sum_i \dot{p}_i dq_i + p_i d\dot{q}_i = \sum_i \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i$$

which is equivalent to

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{3.10}$$

$$\dot{p}_i = \frac{\partial L}{\partial q_i}. \tag{3.11}$$

Eq. (3.10) is the Legendre transformation, and (3.10) substituted in (3.11) gives for a solution of the differential equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \text{ the Euler-Lagrange equations (without external forces).} \tag{3.12}$$

3. Lagrangian submanifolds have a nice interpretation in local coordinates. Let N be a lagrangian submanifold of (M,ω). Because $d\omega = 0$ there exists locally a 1-form α such that $\omega = d\alpha$. N is lagrangian, so $\omega = d\alpha$ restricted to N is zero. From this it follows that (locally) there exists a function V: $N \to \mathbb{R}$ such that $\alpha = dV$ on N. Assume for example that $\omega = \Sigma_i dq_i \wedge dp_i$, $\alpha = \Sigma_i q_i dp_i$ and that N can be parametrized by $p_1, \cdots, p_n$. Then we can see V as a function from $(p_1, \cdots, p_n)$ to $\mathbb{R}$, and N is described (locally) by

$$N = \left\{(p_1, \cdots, p_n, q_1, \cdots, q_n) | q_i = \frac{\partial V}{\partial p_i}\right\}. \tag{3.13}$$

So N is the graph of the function

$$\left(\frac{\partial V}{\partial p_1}, \cdots, \frac{\partial V}{\partial p_n}\right) = dV.$$

We will now state two lemmas which will be used in the following §§ (for a proof see [4])

Lemma 3.5. *Let* φ: $T^*Q \to T^*Q$ *be a diffeomorphism which preserves the fibres of* T^*Q, *and moreover let* φ *be a symplectomorphism, i.e.* $\varphi^*\omega = \omega$ ($\omega = -d\theta$ *where* θ *is the* 1*-form defined by* 3.4), *then there exists a diffeomorphism* ψ: $Q \to Q$ *and a closed* 1*-form* α *on* Q *such that in local coordinates* φ *is given by*

$$(q,p) \underset{\varphi}{\rightarrow} \left(\psi^{-1}(q), \psi^*(p+\alpha)\right).$$

Lemma 3.6. *Let* φ: $T^*Q \to T^*Q$ *be a diffeomorphism which preserves* θ (*see* 3.4), *i.e.* $\varphi^*\theta = \theta$, *then* φ *preserves fibres of* T^*Q *and there exists a diffeomorphism* ψ: $Q \to Q$ *such that* $\varphi = \psi^*$

4. Hamiltonian Systems in State Space Form

For the definition of a Hamiltonian system we need the following extra structure on our differential system in state space form $\Sigma(M, W, B, f)$ (see §2):

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[dr, "\pi"'] & TM \times W \arrow[r, "\pi_2"] \arrow[d, "\pi_M"] \arrow[dr, "\pi_1"] & W \\
& M & TM
\end{tikzcd} \qquad [4.1]$$

(i) M is a manifold with symplectic form ω (2n-dimensional).
(ii) W is a manifold with a symplectic form denoted by ω^e ("e" from external). W is 2m-dimensional.

As we have seen in § 3, the symplectic form ω on M induces a symplectic form on TM, denoted by $\dot{\omega}$. Now we can make $TM \times W$ into a symplectic manifold by defining the symplectic form

$$\Omega := \pi_1^*\dot{\omega} - \pi_2^*\omega^e \quad \text{on } TM \times W.$$

We can now turn to the definition:

Definition 4.1 (a, b).
(*a*) $\Sigma(M, W, B, f)$ with M and W symplectic is called *full Hamiltonian* if f(B) is a lagrangian submanifold of $(TM \times W, \Omega)$

(*b*) $\Sigma(M, W, B, f)$ is called *degenerate Hamiltonian* (M and W are symplectic) if there exists a full Hamiltonian system $\Sigma'(M, W, B', f')$ such that f(B) is a submanifold of f′(B′).

Remark. Definition 4.1 (b) amounts to saying that f(B) is an isotropic submanifold of $(TM \times W, \Omega)$, which has a nice position in $TM \times W$. We call a system $\Sigma(M, W, B, f)$ *autonomous* Hamiltonian if it is autonomous in the sense of def. 2.5 and Hamiltonian in the sense of def. 4.1 (b).

In local coordinates definition 4.1 (a, b) gives the following

Proposition 4.2.
(a) *Let* $\Sigma(M, W, B, f)$ *full Hamiltonian then there exist coordinates* $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ *for* M *and coordinates* $(w_1, \cdots, w_{2m})$ *for* W *and (locally) a function* $H(q_1, \cdots, q_n, p_1, \cdots, p_n, w_1, \cdots, w_m)$ *such that the system is described by*:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, \cdots, n \tag{4.2}$$

and

$$w_{m+j} = c_j \frac{\partial H}{\partial w_j} \qquad j = 1, \cdots, m \tag{4.3}$$

with $c_j = \pm 1$.
(b) *If* $\Sigma(M, W, B, f)$ *is degenerate Hamiltonian, then the system is also locally described by* (4.2) *and* (4.3), *but not all the* w_j, $j = 1, \cdots, m$ *in equation* (4.3) *are free. More exactly if dim* $B = 2n + k$, *with* $k \leq m$, *then* k *of those* w_j*'s are free.*

Remark. (i) The free w_j's with $j \leq m$ in expression (4.2) can be considered as the local inputs. The other w_j's can be considered as the local outputs.
(ii) If the c_j's are not all $+1$, we have what is called in network theory a hybrid representation.

Proof. (a) We know from § 3 that there exist local coordinates

$$(q_1, \cdots, q_n, p_1, \cdots, p_n, \dot{q}_1, \cdots, \dot{q}_n, \dot{p}_1, \cdots, \dot{p}_n) \text{ for TM}$$

and $(\tilde{w}_1, \cdots, \tilde{w}_m, \tilde{w}_{m+1}, \cdots, \tilde{w}_{2m})$ for W such that in these coordinates

$$\Omega = \Sigma_i d\dot{q}_i \wedge dp_i - d\dot{p}_i \wedge dq_i - \Sigma_j d\tilde{w}_j \wedge d\tilde{w}_{m+j}.$$

Because B is a bundle over M and diagram 4.1 commutes, f(B) can be parametrized by $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ and m of the coordinates $\tilde{w}_i$, $i = 1, \cdots, 2m$ of W, not necessarily $\tilde{w}_1, \cdots, \tilde{w}_m$. Call these coordinates $w_1, \cdots, w_m$, and denote the rest of the $\tilde{w}_i$'s by $w_{m+1}, \cdots, w_{2m}$ such that if $w_j = w_k$ for $j \leq m$, then $w_{m+j} = \tilde{w}_{m+k}$ or $\tilde{w}_{k-m}$, dependent on the case that $k \leq m$, or $k > m$.

An easy extension of eq. (3.13) gives us now the the equations (4.2) and (4.3).
(b) If $\Sigma(M, W, B, f)$ is degenerate Hamiltonian we can describe $f(B)$ as a submanifold of a manifold which is locally given by equation (4.2) and (4.3). Because B is a bundle over M and diagram 4.1 commutes we still have full freedom in the $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ variables, but not in the $(w_1, \cdots, w_m)$ variables. □

A very interesting characterization of a Hamiltonian system is given by the following. Take a system $\Sigma \subset K^T$ with K a manifold. Let $k: T \to K$ be an element of Σ (a path in K). We define a variation $\delta k: T \to TK$ as follows: Let $k_n: T \to K$ be a sequence of paths in K ($k_n \in \Sigma$) which converges to k, i.e., $k_n(t) \to k(t)$ for almost every $t \in T$. Then this defines for almost $t \in T$ an element of TK denoted by $\delta k(t)$. Now we can give

Theorem 4.3. *If $\Sigma(M, W, B, f)$ is (full or degenerate) Hamiltonian, then the system $\Sigma \subset (M \times W)^T$ generated by $\Sigma(M, W, B, f)$ satisfies*:

$$\int_{t_1}^{t_2} \omega^e_{w(t)}(\delta_1 w(t), \delta_2 w(t))\,dt = \omega_{x(t_2)}(\delta_1 x(t_2), \delta_2 x(t_2)) - \omega_{x(t_1)}(\delta_1 x(t_1), \delta_2 x(t_1)). \tag{4.4}$$

for all $t_1, t_2 \in T$, *for all* $(x(\cdot), w(\cdot)) \in \Sigma$ *and for all variations* $(\delta_1 x(\cdot), \delta_1 w(\cdot))$ *and* $(\delta_2 x(\cdot), \delta_2 w(\cdot))$ *at* $(x(\cdot), w(\cdot))$ *to* Σ.

Proof. We can consider def. 4.1 (a,b) as the infinitesimal version $(t_1 \to t_2)$ of (4.4). This can be seen as follows.
$x(t_1)$ can be transported to $x(t_2)$. So there is a vectorfield $X: M \to TM$ with the graph of X in $f(B) \subset TM \times W$ projected on TM, which carries $x(t_1)$ over in $x(t_2)$.

Now we deduce (see § 2, eq. 3.5–3.7):

$$X^*\dot{\omega} = X^*(d\bar{\omega}^*\theta) = dX^*(\bar{\omega}^*\theta) \overset{*)}{=} di_X\omega = L_X\omega$$

where at $*)$ we use the general fact that when $\alpha: K \to T^*K$, then $\alpha^*\theta = \alpha$ (θ is the canonical 1-form on T^*K). So we obtain (with some abuse of notation)

$$L_X\omega - \omega^e = 0$$

from which (4.4) readily follows. □

Remark 1. The inverse statement of theorem 4.3 has some difficulties. With the same arguments as in the proof of th. 4.3. we can deduce that if (4.4) holds, then necessarily

$$\pi_1^*\dot{\omega} - \pi_2^*\omega^e = 0 \text{ on } f(B).$$

But, as can be seen from the definition of a Hamiltonian system, this condition is not enough to ensure that $\Sigma(M, W, B, f)$ is Hamiltonian. Only when we also assume that dim (B) = dim M + 1/2. dim W, we know that this condition implies that Σ is full Hamiltonian. It seems that these problems are tied up with such concepts as minimality, local weak controllability and local weak observability (see for the last two concepts Hermann & Krener [11]. In fact we conjecture that when $\Sigma(M, W, B, f)$ is a *minimal* degenerate Hamiltonian system then Σ is *full* Hamiltonian if and only if Σ is in some sense controllable (see for a proof [21]).

Remark 2. Equation (4.4) is a very powerful kind of variational principle. Notice that while most variational principles in the literature (see Leipholz [12]) allow only variations over the q-variables, eq. (4.4) uses variations over the whole set of w-variables. When we assume that these w-variables can be interpreted as the q-variables and the F-variables (the external forces), then we could also allow only variations over the *F-variables*. This seems to correspond to what is called the principle of complementary virtual work (see [12]). All these questions related with eq. (4.4) are under investigation (see also § 9).

Remark 3. It seems useful to define the *energy* of the system given in proposition 4.2. as the function $\tilde{H}(q_1, \cdots, q_n, p_1, \cdots, p_n, w_{m+1}, \cdots, w_{2m})$, which is defined as the Legendre transform of $H(q_1, \cdots, q_n, p_1, \cdots, p_n, w_1 \cdots, w_m)$ with respect to $(w_1, \cdots, w_m)$ (see also [5]). Then we obtain:

$$\frac{d}{dt}\tilde{H}(q_1, \cdots, q_n, p_1, \cdots, p_n, w_{m+1}, \cdots, w_{2m}) = \sum_{i=1}^{m} w_i \frac{d}{dt}(w_{m+i})$$

($\frac{d}{dt}$ denotes time derivative along the trajectories generated by (4.2)). When we interpret $w_1, \cdots, w_m$ as the external forces, and $w_{m+1}, \cdots, w_{2m}$ as the positions, then the right part of this equation equals the instantaneous work exerted on the system. Hence, the change of energy is equal to the external instantaneous work.

5. Lagrangian Systems

Although the Hamiltonian formulation of classical mechanics has the advantage of perfect duality between the q-variables and the p-variables, the Lagrangian formulation in the $(q, \dot{q})$-space is certainly more obvious for most practical problems. Also, phenomena like friction can be better described from the Lagrangian point of view.

Historically the Lagrangian framework was built up with external forces. The general form of the Euler-Lagrange equations is not

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad \text{but} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i,$$

where F_i's are the external forces. We shall see how this fits very naturally in our framework. We start with a system $\Sigma(M, W, B, f)$ (in the sense of def. 2.4) where M and W are symplectic manifolds of a special kind, namely cotangent bundles. So we have $M = T^*Q$ and $W = T^*Y$ (think of Q as the configurationspace, and Y as the outputspace), and on T^*Q is defined the canonical 1-form θ, and on T^*Y is defined the canonical 1-form θ^e (see (3.4)). The 1-form θ on T^*Q induces the 1-form $\dot{\theta}$ on $T(T^*Q)$ (see (3.9)). Now on $T(T^*Q)\times T^*Y$ we can define the 1-form $\Theta := \pi_1^*\dot{\theta} - \pi_2^*\theta^e$ (with π_1 and π_2 the projections of $T(T^*Q)\times T^*Y$ on $T(T^*Q)$, resp. T^*Y). It is clear that $d\Theta$ is a symplectic form on $T(T^*Q)\times T^*Y$. Now we can define, quite analogous to def 3.4, Lagrangian systems as follows.

Definition 5.1. $\Sigma(T^*Q, T^*Y, B, f)$ is called a *Lagrangian* system if $f(B)$ is a lagrangian submanifold of $(T(T^*Q)\times T^*Y, d\Theta)$ and moreover there exists a function $\tilde{L}: TQ\times Y \to \mathbb{R}$ such that Θ restricted to $f(B)$ is equal to $d\tilde{L}$.

Remark. Def. 5.1 implies that a Lagrangian system is a special kind of a Hamiltonian system.

The usual notion of a Lagrangian system we obtain by the following specialization of def. 5.1:

Proposition 5.2. *Let* $\Sigma(T^*Q, T^*Y, B, f)$ *be a Lagrangian system. Let* $(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n)$ *be coordinates for* TQ *and let* $(y_1, \cdots, y_k, F_1, \cdots, F_k)$ *be coordinates for* T^*Y. *Now assume that the outputfunction* $h: TQ \to Y$ *represents the partial observation of the configuration of the system, i.e.,*

$$(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n) \underset{h}{\mapsto} (y_1, \cdots, y_k) = (q_1, \cdots, q_k) \qquad (k \le n).$$

Then we obtain the equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i, \qquad i = 1, \cdots, k$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \qquad i = k+1, \cdots, n. \tag{5.1}$$

Proof. Because $\Sigma(T^*Q, T^*Y, B, f)$ is a Lagrangian system there exists a function $\tilde{L}: TQ\times Y \to \mathbb{R}$ such that, restricted to $f(B)$, $\Theta = d\tilde{L}$. So in coordinates we obtain:

$$\sum_i \dot{p}_i dq_i + p_i d\dot{q}_i - \Sigma_j F_j dy_j = d\tilde{L} \quad \text{on } f(B).$$

Now we define $L(q,\dot{q}) := \tilde{L}(q, \dot{q}, h(q,\dot{q}))$ and use the assumption that $y_j = q_j$, $j = 1, \cdots, k$. Then we obtain

$$\sum_{i=1}^{n} \dot{p}_i dq_i + p_i d\dot{q}_i - \sum_{j=1}^{k} F_j dq_j = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i,$$

and so

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad *)$$

$$\dot{p}_i - \frac{\partial L}{\partial q_i} = F_i \qquad i = 1, \cdots, k \qquad **)$$

$$\dot{p}_i - \frac{\partial L}{\partial q_i} = 0 \qquad i = k+1, \cdots, n.$$

Now substituting $*$) (the Legendre transformation) in $**$) gives eq. (5.1). □

Remark 1. Formulas (5.1) are easily generalized in the case that h: $TQ \to Y$ is of a less special form.

Remark 2. The definition of a Lagrangian system (def. 5.1) has a very historical interpretation. We could interpret the condition

$$\pi_1^* \dot{\theta} - \pi_2^* \theta^e = dL \quad \text{on } f(B)$$

as an abstract formulation of Hamilton's principle, most stated as

$$m\dot{q}\,\delta q\,|_{t_2} - m\dot{q}\,\delta q\,|_{t_1} = \int_{t_1}^{t_2} (\delta T + \delta A_e)\,dt \qquad [5.2]$$

(see [2]), where $|_t$ denotes evaluation on time t, A_e is the external work on the system, and T is the kinetical energy. In the case where there are also (conservative) internal forces, δT becomes δL (with $L = T - V$ and V the potential function which gives the internal forces) and $m\dot{q}$ becomes $\frac{\partial L}{\partial \dot{q}}$.

6. Interconnections of Hamiltonian Systems

As we have already seen in the Introduction (§ 1) interconnection of Hamiltonian systems is frequently encountered.

Definition 6.1 (a,b). Let (W_i, ω_i^e), $i = 1, \cdots, k$ be symplectic manifolds. Then $(W_1 \times \cdots \times W_k, \pi_1^* \omega_1^e + \cdots + \pi_k^* \omega_k^e)$ is a symplectic manifold (π_i are the canonical projections of $W_1 \times \cdots \times W_k$ on W_i).

An *interconnection* of $(W_i)_{i=1,\cdots,k}$ is a submanifold $I \subset W_1 \times \cdots \times W_k$.

(a) An interconnection $I \subset W_1 \times \cdots \times W_k$ is *full Hamiltonian* if I is a lagrangian submanifold of $(W_1 \times \cdots \times W_k, \pi_1^* \omega_1^e + \cdots + \pi_k^* \omega_k^e)$

(b) An interconnection I is called *degenerate Hamiltonian* if I is a co-isotropic submanifold of $(W_1 \times \cdots \times W_k, \pi_1^* \omega_1^e + \cdots + \pi_k^* \omega_k^e)$.

Again we can define a more special version:

Definition 6.2. Let (T^*Y_i, θ_i^e), $i=1,\cdots,k$ be cotangent bundles, with θ_i^e the canonical 1-forms. Then $(T^*Y_1 \times \cdots \times T^*Y_k, \pi_1^*\theta_1^e + \cdots + \pi_k^*\theta_k^e)$ is a special kind of a symplectic manifold. The symplectic form is $\Omega := d(\pi_1^*\theta_1^e + \cdots + \pi_k^*\theta_k^e)$. An interconnection $I \subset T^*Y_1 \times \cdots \times T^*Y_k$ is (full or degenerate) *Lagrangian* if I is (full or degenerate) Hamiltonian and moreover $\pi_1^*\theta_1^e + \cdots + \pi_k^*\theta_k^e$ restricted to I is exact.

Remark. We have defined full Hamiltonian (Lagrangian) interconnections as memoryless (i.e. without state space) Hamiltonian (Lagrangian) systems.

In practice we encounter mostly a further specialized version of a Hamiltonian interconnection. Because of its intimate relation with the Kirchhoff's laws in network theory we state separately

Definition 6.3. A Lagrangian interconnection I is called a Kirchhoff interconnection if $\pi_1^*\theta_1^e + \cdots + \pi_k^*\theta_k^e$ restricted to I is zero.

To see what all these definitions amount to, we begin with the Kirchhoff interconnection and assume for simplicity that we have only two manifolds $W_1 = T^*Y_1$ and $W_2 = T^*Y_2$. We have local coordinates such that $\theta_1^e = \sum_{j=1}^{m_1} u_i^1 dy_i^1$ and $\theta_2^e = \sum_{j=1}^{m_2} u_j^2 dy_j^2$. Assume further $m_1 \geq m_2$ and that the interconnection $I \subset W_1 \times W_2$ is full and can be parametrized by (y_i^1, u_j^2), $i=1,\cdots,m_1$, $j=1,\cdots,m_2$. Then we state

Proposition 6.4. *I is full Kirchhoff interconnection* $\Leftrightarrow$ *there exists a function* φ: $Y_1 \to Y_2$ *such that*

$$I = \{(y^1, u^1, y^2, u^2) \in T^*Y_1 \times T^*Y_2 \,|\, y^2 = \varphi(y^1),\ \varphi^*(u^2) = -u^1\}.$$

(*The proof of this is in fact an extension of lemma* 3.6.)

So we have here that I is, what is called in network theory, *nonmixing*, i.e., y^2 is only related to y^1, u^2 is only related to u^1. Moreover, we see that the relation between u^2 and u^1 is *linear*.

An even more special case is when $W_1 = Y_1 \times U_1$ and $W_2 = Y_2 \times U_2$, and when not only $\sum_{i=1}^{m_1} u_i^1 dy_i^1 + \sum_{j=1}^{m_2} u_j^2 dy_j^2$ but also the dual 1-form $\sum_{i=1}^{m_1} y_i^1 du_i^1 + \sum_{j=1}^{m_2} y_j^2 du_j^2$ are zero restricted to I. It follows from the proposition above that the Kirchhoff interconnection in this case is necessarily (total) linear, i.e. there exists a linear map A such that $y^2 = Ay^1$ and $u^1 = -A^T u^2$. These are exactly Kirchhoff's laws! (see also Brayton [13], Hermann [14])

When the interconnection is only Lagrangian we obtain the following

Proposition 6.5. I *is a full Lagrangian interconnection iff there exists* φ: $Y_1 \to Y_2$ *and* (*locally*) V: $Y_1 \to \mathbf{R}$ *such that*

$$I = \{(y^1,u^1,y^2,u^2)\in T^*Y_1 \times T^*Y_2 \mid y^2=\varphi(y^1),\ u^1+dV(y^1)=\varphi^*(u^2)\}.$$

(*This is in fact an extension of lemma* 3.5.)

As we would expect, interconnection of Hamiltonian systems results in another Hamiltonian system.

Theorem 6.6. *Let* $\Sigma_i(M_i,W_i,B_i,f_i)$, $i=1,\cdots,k$ *be Hamiltonian systems, interconnected by a Hamiltonian interconnection* $I\subset W_1\times\cdots\times W_k$. *The resulting system is a Hamiltonian system* $\Sigma_I(M_1\times\cdots\times M_k,W_1\times\cdots\times W_k,B_I,f_I)$.

Proof. (sketch). We can construct the *product system*, denoted by $\Sigma(M_1\times\cdots\times M_k,W_1\times\cdots\times W_k,B_\times,f_\times)$, of $\Sigma_i(M_i,W_i,B_i,f_i)$, $i=1,\cdots,k$ as follows. Let $x=(x_1,\cdots,x_k)\in M_1\times\cdots\times M_k$. Because π_i: $B_i\to M_i$ are fibre bundles there exist neighborhoods $U_i\subset M_i$ of x_i such that $\pi_i^{-1}(U_i)\cong U_i\times F_i$, where F_i is the so called standard fibre. Now define $B_\times$ locally as $\pi_\times$: $(U_1\times\cdots\times U_k)\times(F_1\times\cdots\times F_k)\to U_1\times\cdots U_k$. Next define $f_\times$: $B_\times\to T(M_1\times\cdots\times M_k)\times(W_1\times\cdots\times W_k)$ locally as $f_\times:=(f_1,\cdots,f_k)$. Then it is easy to see that when $\Sigma_i(M_i,W_i,B_i,f_i)$ are Hamiltonian then $\Sigma(M_1\times\cdots\times M_k,W_1\times\cdots\times W_k,B_\times,f_\times)$ is Hamiltonian (with the usual symplectic forms on $M_1\times\cdots\times M_k$ and $W_1\times\cdots\times W_k$). Construct B_I and f_I such that restricted to $W_1\times\cdots\times W_k$ $f_I(B_I)=f_\times(B_\times)\cap I$ (of course we assume that $f_\times(B_\times)\cap I$ is again a manifold). Now it is clear that the interconnected system $\Sigma_I(M_1\times\cdots\times M_k,W_1\times\cdots\times W_k,B_I,f_I)$ satisfies $f_I(B_I)\subset f_\times(B_\times)$ and so by def. 4.1 $\Sigma_I(M_1\times\cdots\times M_k,W_1\times\cdots\times W_k,f_I,B_I)$ is a Hamiltonian system. □

Remark 1. Usually interconnection of Hamiltonian systems results in a Hamiltonian system which is more degenerate then the original systems. For example, when $\Sigma_i(M_i,W_i,B_i,f_i)$ are full Hamiltonian systems, and $I\subset W_1\times\cdots\times W_k$ is a full Hamiltonian interconnection then $\Sigma_I(M_1\times\cdots\times M_k,W_1\times\cdots\times W_k,B_I,f_I)$ satisfies $\dim f_I(B_I)=\dim M_1\times\cdots\times M_k$, so Σ_I is nearly an *autonomous* system (see def. 2.5)

Remark 2. If full Hamiltonian systems $\Sigma_i(M_i,W_i,B_i,f_i)$ are interconnected by a degenerate Hamiltonian interconnection one could ask if it is possible to *reduce* the space of external variables $W_1\times\cdots\times W_k$ such that the interconnected system is full Hamiltonian w.r.t. to this reduced external space. Indeed, following Weinstein [15], one can show that if I is co-isotropic then, at least locally, this is possible.

7. Realization Theory for Hamiltonian Systems

An important topic in mathematical systems theory is the realization problem. The central question is how we can derive from a system in external form (see § 2) a system in state space form which has the same external behavior as the original system. A second, related, question is how the structure of the external system can be mirrored in the structure of the state space system. In the case of Hamiltonian systems we should like to know which conditions the behavior of the external system has to satisfy in order that it is possible to construct a Hamiltonian realization, i.e. a system in state space form as defined in def. 4.1 (a, b). Because a realization theory for nonlinear systems is still in development, we will look first at the special case of linear Hamiltonian systems (see also the Conclusion).

While a realization theory for linear input-output systems is well-known (see [16]), the theory for the more general linear systems as defined in § 2 is new. Because a full treatment of this subject is out of the scope of this paper, we will state only some results. Linear external systems Σ_e are of the following form:

$$\Sigma_e = \left\{ w: T \to W \,|\, P\left(\frac{d}{dt}\right) w(t) = 0 \right\}$$

where W is a linear space and P is a polynomial matrix. After Laplace transformations we obtain the following system in *frequency* domain: $\Sigma_e^f = \{w: S^2 \to W_C \,|\, P(s)w(s) = 0 \ \forall s \in S^2\}$, where S^2 is the Riemannsphere and W_C the complexification of W. So we can look at our system as a "fibre bundle" over S^2 with fibre at s the kernel of P(s) (the dimension of the fibres is not necessarily constant). A *realization* of Σ_e consists of a 4-tuple $\{A, B, C, D\}$ such that the external behavior of the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ w &= Cx + Du \end{aligned} \qquad [7.1]$$

is equal to Σ_e (u is the input). One can derive the following (see Willems [7])

Theorem 7.1.

(a) *Let* $\Sigma_i = \{A_i, B_i, C_i, D_i\}$, $i = 1, 2$, *both be minimal realizations of* Σ_e; *then there exists a unique nonsingular matrix* S *such that*

$$\{(x(\cdot), w(\cdot))\} \in \Sigma_1 \Leftrightarrow \{Sx(\cdot), w(\cdot)\} \in \Sigma_2$$

(b) *All minimal realizations* $\{A, B, C, D\}$ *of* Σ_e *with rank* $\begin{pmatrix} B \\ D \end{pmatrix} = m$ *may be obtained from any minimal realization* $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ *with rank* $\begin{pmatrix} \bar{B} \\ \bar{D} \end{pmatrix} = m$ *by the feedback group*

$$\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\} \underset{\substack{\det S \neq 0 \\ \det R \neq 0}}{\overset{F,S,R}{\longmapsto}} \{S(\bar{A} + \bar{B}F)S^{-1}, S\bar{B}R, (\bar{C} + \bar{D}F)S^{-1}, \bar{D}R\}.$$

We now turn to the realization of linear Hamiltonian systems.

Proposition 7.2. *A linear system* $\Sigma = \{A, B, C, D\}$ *is (full or degenerate) Hamiltonian (see def.* 4.1(*a*, *b*)) *iff*

$$\begin{aligned} A^TJ + JA &= 0 \\ B^TJ + D^TJ_eC &= 0 \\ C^TJ_eC &= 0 \\ D^TJ_eD &= 0 \end{aligned} \qquad [7.2]$$

where J *and* J_e *are linear symplectic forms on the linear spaces* M *resp.* W. Σ *is full (degenerate) Hamiltonian iff dim* $D = 1/2$. *dim* W (*dim* $D \le 1/2$ *dim* W).

Proof. Write out def. 4.1 (a, b) for the system $\dot{x} = Ax + Bu$, $w = Cx + Du$ and note that you may apply feedback to obtain (7.2) □

We now look for necessary and sufficient conditions on the external system, i.e. P(s), in order that it is possible to realize P(s) with matrices $\{A, B, C, D\}$ satisfying (7.2). Therefore we introduce on W_C a symplectic form ω_C^e as follows:

$$\omega_C^e(w_1, w_2) := w_1^* J_e w_2, \qquad [7.3]$$

where J_e is a symplectic form on W and $*$ denotes the Hermitian conjugate. First we give for the full Hamiltonian case the following theorem which we state without proof:

Theorem 7.3. *If Ker* P(s) *is lagrangian with respect to* ω_C^e *for all* $s = i\omega$, $\omega \in \mathbb{R}$, *then there exists a minimal realization which is full Hamiltonian. Moreover this realization is necessarily controllable.*

For the degenerate Hamiltonian case, we state (the easily proven).

Theorem 7.4. P(s) *can be realized by a degenerate Hamiltonian system* $\{A, B, C, D\}$ *iff there exists a matrix* $\tilde{P}(s)$ *satisfying the conditions of theorem* 7.3 *and a co-isotropic matrix* K (*i.e.* $KJ_eK^T = 0$) *such Ker* P(s) = *Ker* $\begin{pmatrix} \tilde{P}(s) \\ K \end{pmatrix}$.

Remark 1. K can be interpreted as a degenerate Hamiltonian interconnection. So a degenerate linear Hamiltonian system is really a full Hamiltonian system with some extra internal Hamiltonian interconnections.

Remark 2. A linear system in input-output form is of the form:

$$\Sigma_e = \left\{ (y(t), u(t)) | Q\left(\frac{d}{dt}\right) y(t) = R\left(\frac{d}{dt}\right) u(t) \right\}$$

or in frequency domain

$$\Sigma_e^f = \{ (y(s), u(s) | Q(s)y(s) = R(s)u(s) \}.$$

We can define the transfer function $G(s) := Q^{-1}(s)R(s)$, so $y(s) = G(s)u(s)$. The condition on $P(s) := [Q(s) : -R(s)]$ of theorem 7.3 is equivalent to $G(s) = G^T(-s)$.

Then G(s) has a Hamiltonian realization in input-outputform, given by

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du \quad \text{with } A^T J + JA = 0,\ B^T J = C \quad \text{and} \quad D = D^T$$

(see for instance [17], [18]). This is easily seen to be a direct specialization of equations (7.2).

8. Symmetries

An important subject in the study of mechanical systems consists of the so-called symmetries. These are (sometimes infinitesimal) deformations of the state space (i.e. the phase space) which leave—loosely speaking—the system invariant. The existence of such symmetries gives more insight in the structure of the system. At the same time the symmetries can be used to derive extra integrals for the system —i.e. functions which are constant on the integral curves—, and so to really integrate the differential equations which constitute the system. Our approach will be to derive such symmetries on the state space from symmetries on the external systems. First we shall define symmetries for general (not necessarily Hamiltonian) systems.

Definition 8.1. A symmetry for an *external* system $\Sigma \subset W^T$ (see § 2) is a map ψ: $W \to W$ which leaves Σ invariant; i.e. if $w(\cdot) \in \Sigma$, then also $\psi(w(\cdot)) \in \Sigma$, and if $w(\cdot) \in \Sigma$ then there exists $\tilde{w}(\cdot) \in \Sigma$ such that $\psi(\tilde{w}(\cdot)) = w(\cdot)$.

Definition 8.2. A symmetry for a system in *state space form* $\Sigma \subset X^T \times W^T$ is a map (φ, ψ): $X \times W \to X \times W$ which leaves Σ invariant; i.e. if $(x(\cdot), w(\cdot)) \in \Sigma$ then also $(\varphi(x(\cdot), \psi(w(\cdot))) \in \Sigma$, and if $(x(\cdot), w(\cdot)) \in \Sigma$ then there exists $(\tilde{x}(\cdot), \tilde{w}(\cdot)) \in \Sigma$ such that $(\varphi(\tilde{x}(\cdot)), \psi(\tilde{w}(\cdot))) = (x(\cdot), w(\cdot))$

Now we turn to differential systems, for which we specialize def. 8.2 to

Definition 8.3. A symmetry for a differential system in state space form $\Sigma(M, W, B, f)$ is a 3-tuple (φ, ψ, Φ) such that φ: $M \to M$, ψ: $W \to W$, Φ: $B \to B$ are diffeomorphisms for which the following diagram

$$\begin{tikzcd}
B \arrow[rrr, "\Phi"] \arrow[dr, "f"] \arrow[dddd, "\pi"'] & & & B \arrow[dl, "f"'] \arrow[dddd, "\pi"] \\
& W \arrow[r, "\psi"] & W & \\
& \times & \times & \\
& TM \arrow[r, "\varphi_*"'] \arrow[dl, "\pi_M"] & TM \arrow[dr, "\pi_M"'] & \\
M \arrow[rrr, "\varphi"'] & & & M
\end{tikzcd}$$

commutes. [8.1]

Remark . Also an analogous definition for *infinitesimal* symmetries can be given.

From now on we will look only at symmetries for Hamiltonian systems. The concept of symmetry becomes stronger in this case because we want to have that the symmetry preserves the extra, symplectic, structure.

Definition 8.4. A symmetry of a Hamiltonian system is a 3-tuple (φ, ψ, Φ) such that the conditions of def. 7.3 are satisfied and moreover $\psi^*\omega^e = \omega^e$ and $\varphi^*\omega = \omega$.

Remark. $\varphi^*\omega = \omega$ is equivalent with $(\varphi_*)^*\dot{\omega} = \dot{\omega}$.

After establishing this general framework we will work out the linear Hamiltonian case. Definition 8.4 easily specializes to

Definition 8.5. A *symmetry* of a *linear* Hamiltonian system $\{A, B, C, D\}$ is a 4-tuple (Q, S, H, R) such that: (Q: $W \to W$, S: $M \to M$, H: $W \to X$, R:$U \to U$)

(i) $Q^T J_e Q = J_e$, $S^T J S = J$

(ii) $A = S(A + BHC)S^{-1}$

$B = SBR$

$C = Q(C + DHC)S^{-1}$

$D = QDR$.

Remark 1. Notice that the feedback has the form HC; i.e, *output*-feedback. This follows from diagram (8.1) specialized to the linear case.

Remark 2. Because S is a symplectomorphism ($S^T J S = J$) and A is Hamiltonian, it follows from $A = S(A + BHC)S^{-1}$ that also BHC is Hamiltonian, or equivalently (if B is injective) that H is symmetric. We can state the following

Theorem 8.6. *Let Q be an external symmetry for a full Hamiltonian system* P(s); *i.e.* P(s) *satisfies the conditions of theorem* 7.3, *and*

(i) *Ker* P(s) = *Ker* P(s)Q,

(ii) $Q^T J_e Q = J_e$,

then there exist (S, H, R) *such that* (Q, S, H, R) *is a symmetry for a minimal Hamiltonian realization* {A, B, C, D} *of* P(s). *Moreover, when* Q *leaves Im*C *invariant, then* H *can be taken equal to zero.*

Proof. Let {A, B, C, D} be a minimal Hamiltonian realization of P(s). Then because Ker P(s) = Ker P(s)Q also {A, B, QC, QD} is a minimal Hamiltonian realization of P(s). Applying theorem 7.1 gives the result. □

Remark 1. In fact we could state a stronger result: if we have a group of symmetries Q on W, we can easily deduce that we also obtain a group of symmetries S on the state space.

Remark 2. Some related results on symmetries can be found in [19].

The framework sketched above is further elaborated in [22].

9. Conclusion

We have given definitions of Hamiltonian and Lagrangian systems with external variables, which, in our opinion, are very natural and which fit easily in notions like Hamiltonian vectorfields, variational principles, interconnections and symmetries. Of course a lot of work remains to be done. What is most needed from the system theoretic point of view is a realization theory for general nonlinear Hamiltonian systems. The linear theory as described in § 7 gives some information how this might go. Also formula (4.4) of theorem 4.3.

$$\int_{t_1}^{t_2} \omega^e_{w(t)}(\delta_1 w(t), \delta_2 w(t))\, dt = \omega_{x(t_2)}(\delta_1 x(t_2), \delta_2 x(t_2)) - \omega_{x(t_1)}(\delta_1 x(t_1), \delta_2 x(t_1))$$

seems to provide a good starting point for such a theory because the left side contains only external terms. This is presently under investigation.

Applications of the general theory developed in this paper can be sought for instance in optimization theory, which, by Pontryagin's maximum-principle, is closely related to our framework. From a system theoretic viewpoint classical *thermodynamics* is a very intriguing subject. It is very natural to see the pressure (P), the volume (V), the temperature (T) and possibly the heat flow (Q) as external variables. In particular it is clear that it is not natural to assume one variable as the cause of another, but only to assume a compatibility relation like (for ideal gases) $PV=cT$ with c a constant.

One could try to construct from such external relations a state space model, in terms of which the (internal) energy and the entropy are defined. From a historical point of view (see [1]) it would be interesting to investigate how far it is possible to construct a *mechanical* model for thermodynamic systems, where mechanical is interpreted in a broad sense (not only Hamiltonian but also some sort of *gradient* behavior). Already an attempt in this direction is made in [7, 20].

The same ideas applied in this paper to Hamiltonian systems can be used for describing a broader class of physical systems. For instance, *gradient systems* fit also naturally in this scheme. Analogous to § 1.2 we can see standard gradient systems in a mechanical context as follows:

System I. $m\dot{v}_1 = F_1$ where v_1 is the velocity, F_1 the external force, m is the mass, which in more general gradient systems is replaced by a Riemannian metric.

System II. $F_2 = \dfrac{dR}{dv_2}$ which describes the force due to a friction dependent on the velocity. Systems I and II can again be interconnected by setting

$$v_1 = v_2 \qquad F_1 = F_2.$$

So we can define a general gradient system by taking W a symplectic manifold and M a manifold with a (not necessarily positive definite) Riemannian metric

$\langle , \rangle$. Because $\langle , \rangle$ is nondegenerate we can again define a bundle isomorphism α: $TM \to T^*M$ by setting $\alpha(X) := \langle X, - \rangle$ for $X \in TM$. T^*M has a canonical 2-form ω, so $\alpha^*\omega$ is a symplectic form on TM (compare with the definition of $\dot{\omega}$). Then $TM \times W$ has symplectic form $\Omega := \pi_1^*\alpha^*\omega - \pi_2^*\omega^e$, with π_1 and π_2 the projections of $TM \times W$ on TM, resp. W and ω^e the symplectic form on W. We call $\Sigma(M, W, B, f)$ a full *gradient system* if f(B) is a lagrangian submanifold of $(TM \times W, \Omega)$.

Most of the results obtained in §§ 6, 7, 8 can also easily be deduced for such gradient systems.

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